Induced Representations

Jackson Gatenby

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1 Representations of a Group Product

We first take an aside to consider representations formed by taking a group product. Let G_1, G_2 be groups, and let $\rho_1 : G_1 \to GL(V_1)$ and $\rho_2 : G_2 \to GL(V_2)$ be representations of G_1, G_2 on V_1, V_2 respectively. We can define the tensor product $\rho_1 \otimes \rho_2 : G_1 \times G_2 \to GL(V_1 \otimes V_2)$ by

$$(\rho_1 \otimes \rho_2)(g_1, g_2) = \rho_1(g_1) \otimes \rho_2(g_2),$$

(recall that for $f \in GL(V_1), g \in GL(V_2)$, $f \otimes g \in GL(V_1 \otimes V_2)$ is given by $v_1 \otimes v_2 \mapsto f(v_1) \otimes g(v_2)$).

We have that $\rho_1 \otimes \rho_2$ is a linear representation of $G_1 \otimes G_2$; the proof is similar to our previous definition of a tensor product of representations: for $g_i, h_i \in G_i$, $v_i \in V_i$ $(i = 1, 2)$, we have

$$\begin{aligned}(g_1h_1, g_2h_2) \cdot (v_1 \otimes v_2) &= (g_1h_1 \cdot v_1) \otimes (g_2h_2 \cdot v_2)\\ &= (g_1, g_2) \cdot (h_1 \cdot v_1 \otimes h_2 \cdot v_2)\\ &= (g_1, g_2)(h_1, h_2) \cdot (v_1 \otimes v_2),\end{aligned}$$

and by linearity, the result extends to all of $V_1 \otimes V_2$. (Here, we used the notation $g \cdot v$ to mean $\rho(g)v$ where the representation ρ is understood.) □

Our former definition of a representation tensor product is a particular case of this: if we take the case where $G = G_1 = G_2$, and restrict the representation to the diagonal subgroup $\{(g, g) \in G \times G \mid g \in G\} \simeq G$, we obtain our previous definition. Note however that we should still distinguish between a representation $\rho_1 \otimes \rho_2$ on G and on $G \times G$, despite the identical notation.

If χ_1, χ_2 are the characters of ρ_1, ρ_2, then we also have that the character χ of $\rho_1 \otimes \rho_2$ is also given by

$$\chi(g_1, g_2) = \chi_1(g_1)\chi_2(g_2).$$

We then have the following result:

Theorem 1. *If ρ_1, ρ_2 are irreducible representations of G_1, G_2 on V_1, V_2 respectively, then $\rho_1 \otimes \rho_2$ is an irreducible representation of $G_1 \times G_2$. Moreover, each irreducible representation on $G_1 \times G_2$ is isomorphic to $\rho_1 \otimes \rho_2$ for some irreducible ρ_1, ρ_2 of G_1, G_2.*

Hence, if a group G can be decomposed as $G = G_1 \times G_2$, this reduces the study of representations on G to representations on G_1 and G_2.

Proof. If ρ_1 and ρ_2 are irreducible with characters χ_1, χ_2, and $\chi_{1,2}$ is the character of $\rho_1 \otimes \rho_2$, then

$$\begin{aligned}
(\chi_{1,2} \mid \chi_{1,2}) &= \frac{1}{|G_1 \times G_2|} \sum_{\substack{g_1 \in G_1 \\ g_2 \in G_2}} |\chi_{1,2}(g_1, g_2)|^2 \\
&= \frac{1}{|G_1||G_2|} \sum_{\substack{g_1 \in G_1 \\ g_2 \in G_2}} |\chi_1(g_1)\chi_2(g_2)|^2 \\
&= \frac{1}{|G_1|} \left(\sum_{g_1 \in G_1} |\chi_1(g_1)|^2 \right) \cdot \frac{1}{|G_2|} \left(\sum_{g_2 \in G_2} |\chi_2(g_2)|^2 \right) \\
&= (\chi_1 \mid \chi_1)(\chi_2 \mid \chi_2) = 1 \cdot 1 = 1,
\end{aligned}$$

whence $\chi_{1,2}$ is irreducible.

To show that each irreducible representation of $G_1 \times G_2$ is isomorphic to some $\rho_1 \otimes \rho_2$, Serre gives the following proof: it suffices to show that for all characters of the form $\chi_{1,2}(g_1, g_2) = \chi_1(g_1)\chi_2(g_2)$ for irreducible χ_1, χ_2 of G_1, G_2, that any class function f on $G_1 \times G_2$ which is orthogonal to each $\chi_{1,2}$ is identically zero, and so the $\chi_{1,2}$ form an orthonormal basis for the space of class functions. Suppose then that

$$\begin{aligned}
0 = (f \mid \chi_{1,2}) &= \sum_{\substack{g_1 \in G_1 \\ g_2 \in G_2}} f(s_1, s_2)\overline{\chi_1(s_1)\chi_2(s_2)} \\
&= \sum_{g_1 \in G_1} \left(\sum_{g_2 \in G_2} f(g_1, g_2)\overline{\chi_2(g_2)} \right) \overline{\chi_1(g_1)}.
\end{aligned}$$

Letting $\hat{f}(g_1) := \sum_{g_2 \in G_2} f(g_1, g_2)\overline{\chi_2(g_2)}$, we then have $(\hat{f} \mid \chi_1) = 0$, so $\hat{f} \equiv 0$ (since it is easily seen that $\hat{f}$ is a class function on G_1). Then for any $g_1 \in G_1$,

$$0 = \hat{f}(g_1) = \sum_{g_2 \in G_2} f(g_1, g_2)\overline{\chi_2(g_2)} = (f(g_1, \cdot) \mid \chi_2),$$

and since $f(g_1, \cdot)$ is a class function on G_2, we have that $f(g_1, \cdot) \equiv 0$ for all g_1, hence $f \equiv 0$. □

2 Induced Representations

Let G be a group, and let H be a subgroup of G. If we have a representation W of H, can we 'extend' it to form a unique representation of G?

Let $\rho : G \to GL(V)$ be a representation, and let $W \leq V$ be a subspace which is H-invariant (that is, W is a subrepresentation of the restriction $\rho|_H : H \to GL(V)$. We can also denote a restriction of ρ to H by $\mathrm{Res}_H^G V$, or simply $\mathrm{Res}\, V$). Then, for any $g \in G$, $h \in H$, we have

$$\rho(gh)W = \rho(g) \circ \rho(h)W = \rho(g)W,$$

hence the collection $\{\rho(g)W \mid g \in G\}$ is indexed by the left cosets of G/H; for each left coset $\sigma \in G/H$, we can write $\sigma \cdot W$ or W_σ for $\rho(g)W$ (where $\sigma = gH$). Moreover, each $g \in G$ permutes these translates of W, so that $\sum_{\sigma \in G/H} \sigma \cdot W$ is a subrepresentation of V.

If we were given G, H, and the action of H on W, under what conditions could we uniquely reconstruct, or 'induce' the action of G on V?

Definition 1. We say that the representation V of G is *induced* by W if

$$V = \bigoplus_{\sigma \in G/H} \sigma \cdot W.$$

Lemma 1. *Let G be a group, H a subgroup of G, and W be a representation of H. There exists a representation (ρ, V) of G which is induced by W.*

Proof. We give a construction of V: For each $\sigma \in G/H$, construct a copy W_σ of W. Select a representative $g_\sigma \in \sigma$ (for $\sigma = H$, choose $g_\sigma = e$), and let $g_\sigma w \in W_\sigma$ denote the copy of $w \in W$. Then, let $V = \bigoplus_{\sigma \in G/H} W_\sigma$.

It remains to define a representation action ρ on V, we take the natural choice: For $s \in G$, we define $\rho(s)(g_\sigma w)$ as follows: let $\tau = s\sigma \in G/H$ with representative g_τ, so that $h := g_\tau^{-1} s g_\sigma \in H$. Then, set

$$\rho(s)(g_\sigma w) = g_\tau(\rho_H(h)w),$$

and extend linearly to all of V. In particular, $\rho(h)w = \rho_H(h)w$ for $h \in H, w \in W$. To see that this is a representation on V, we must show that $\rho(s's)(g_\sigma w) = \rho(s')\rho(s)(g_\sigma w)$. Let $\tau' = s'\tau \in G/H$ and $h' = g_{\tau'}^{-1} s' g_\tau \in H$. Then,

$$\begin{aligned} \rho(s's)(g_\sigma w) &= g_{\tau'}(\rho_H(h'h)w) \\ &= g_{\tau'}(\rho_H(h')\rho_H(h)w) \\ &= \rho(s')(g_\tau(\rho_H(h)w)) \\ &= \rho(s')\rho(s)(g_\sigma w). \end{aligned}$$

By linearity, we have that ρ forms a representation on V. □

To see uniqueness of the induced representation, we shall demonstrate the following:

Lemma 2. *Let W be a representation of H, and suppose that V is induced by H. Then, for any representation U of G, and map $\phi : W \to U$, there exists a unique representation map $\tilde{\phi} : V \to U$ such that $\tilde{\phi}(w) = \phi(w)$ for all $w \in W$. Equivalently,*

$$Hom_H(W, Res\, U) = Hom_G(Ind\, W, U).$$

Proof. Suppose $V = \bigoplus_{\sigma \in G/H} \sigma \cdot W$. By linearity, it suffices to show the result for arbitrary $\sigma \in G/H$ and $x \in \sigma \cdot W$. Suppose $\tilde{\phi}$ satisfies this condition, then for any $g \in \sigma$, we must have $g^{-1}x \in W$, hence

$$\tilde{\phi}(x) = \tilde{\phi}(gg^{-1} \cdot x) = g \cdot \tilde{\phi}(g^{-1} \cdot x) = g \cdot \phi(g^{-1} \cdot x).$$

This uniquely determines $\tilde{\phi}$; moreover we can define $\tilde{\phi}$ in this way: take an arbitrary $g \in \sigma$, and let

$$\tilde{\phi}(x) := g \cdot \phi(g^{-1} \cdot x).$$

This map is independent of the choice of g; if $g = gh$ for $h \in H$, then

$$gh \cdot \phi((gh)^{-1} \cdot x) = g \cdot h \cdot \phi(h^{-1}g^{-1} \cdot x) = g \cdot \phi(hh^{-1}g^{-1} \cdot x) = g \cdot \phi(g^{-1} \cdot x).$$

It is readily verified that $\tilde{\phi}$ is a representation map. □

We therefore have the following result:

Theorem 2. *Let G be a group, H a subgroup of G, and W be a representation of H. There exists a unique (up to isomorphism) representation V of G which is induced by W.*

Proof. Existence is given in Lemma 1. For uniqueness, suppose U and V are both induced by W, then Lemma 2 gives an isomorphism between them. □

If V is induced by W, we write $V = \mathrm{Ind}_H^G W = \mathrm{Ind}\, W$.

We can now consider the character of $V = \mathrm{Ind}\, W$. Note that $g \in G$ maps σW to $g\sigma W$, so the trace of $\chi_V(g)$ is given by only those cosets where $g\sigma = \sigma$. For arbitrary $s \in \sigma$, since $\rho(s) : W \to \sigma \cdot W$ is an isomorphism with $\rho(s) \circ \rho(s^{-1}gs)|_W = \rho(g)|_{\sigma \cdot W} \circ \rho(s)$, we have that $tr(\rho(g)|_{\sigma \cdot W}) = tr(\rho(s^{-1}gs)|_W) = \chi_W(s^{-1}gs)$. Since V is the direct sum of all $\sigma \cdot W$, we have

$$\chi_V(g) = \sum_{g\sigma = \sigma} \chi_W(s^{-1}gs) = \frac{1}{|H|} \sum_{\substack{s \in G \\ s^{-1}gs \in H}} \chi_W(s^{-1}gs).$$

In the first equality, we take an arbitrary representative $s \in \sigma$ for each σ. The second equality follows from noting that $\chi_W(s^{-1}gs) = \chi_W(r^{-1}gr)$ when $sH = rH$.

2.1 An Alternative Definition

We also present an alternative, analytic definition of the induced representation of W:

Definition 2. Let G be a group, and let H be a subgroup of G. If (π, W) is a representation of H, let

$$V = \{f : G \to W \mid \forall g \in G, h \in H, f(hg) = \pi(h)f(g)\}$$

For $g \in G$, define an action $\rho(g)$ on V which sends $f \in V$ to the function $\rho(g)f$, where

$$(\rho(g)f)(x) = f(xg).$$

Then, ρ is a representation on V, and we say that V is the *induced representation* of W.

It remains to show that ρ is indeed a representation, and that this definition agrees with Definition 1. It may be an interesting exercise to give a proof of Lemma 2 for this definition.

3 Examples

We end by listing some examples and other results. Checking these is left as an exercise.

- Take V to be the regular representation of G, with basis $\{e_g \mid g \in G\}$, and take W to be the regular representation of H. Then $V = \text{Ind } W$.
- Take V to be the permutation representation of G associated with G/H, with basis $\{e_\sigma \mid \sigma \in G/H\}$, and take W to be the trivial subrepresentation $\mathbb{C}e_H$. Then $V = \text{Ind } W$.
- If $W = \bigoplus W_i$, then $\text{Ind } W = \bigoplus \text{Ind } W_i$.
- If U is a representation of G, and W a representation over H, then $U \otimes \text{Ind } W = \text{Ind}(\text{Res}(U) \otimes W)$. In particular, $\text{Ind}(\text{Res}(U)) = U \otimes P$, where P is the permutation representation associated with G/H. A description of $\text{Res}(\text{Ind}(W))$ is given in Serre §7.3.
- If $H \leq K \leq G$, then $\text{Ind}_H^G(W) = \text{Ind}_K^G(\text{Ind}_H^K(W))$.

References: Serre §3.2, 3.3; Fulton & Harris §3.3.